High-Frequency Trading in a Limit Order Book

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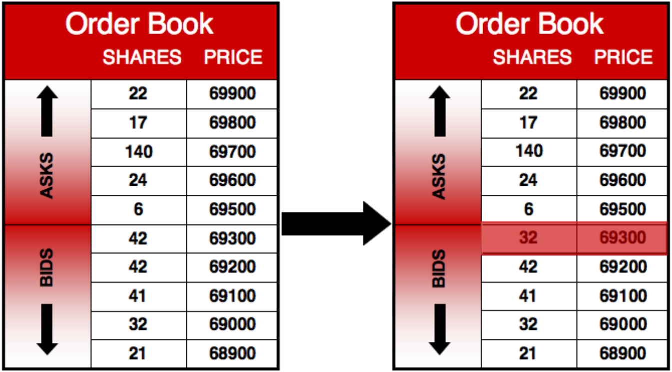
The limit order book



Order Book

	SHARES	PRICE
ASKS	22	69900
	17	69800
	140	69700
	24	69600
	6	69500
BIDS	42	69300
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	41	69100
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  1. Liquidity taker: buys at the ask, sell at the bid
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     - The stock volatility: σ
     - The risk aversion: γ
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  2. Adverse selection risk

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1. Optimization
   - The maximal utility problem
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   - Some approximations
   - P&L profiles of the optimal strategy

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4. Conclusion

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- Brownian motion

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- Trading at the mid-price is not allowed. However, we may quote limit orders p^b and p^a around the mid-price.

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- The wealth in cash

$$dX_t = p^a dN_t^a - p^b dN_t^b$$

The inventory

$$q_t = N_t^b - N_t^a$$

The market maker's objective

- Maximize exponential utility

$$u(s, x, q, t) = \max_{p_t^a, p_t^b, 0 \le t \le T} E_t \left[-e^{-\gamma(X_T + q_T S_T)} \right]$$

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- Infinite horizon exponential utility

$$w(x,s,q) = \max_{p_t^a, p_t^b} E \left[\int_0^\infty -\exp(-\omega t) \exp(-\gamma (X_t + q_t S_t)) dt \right].$$

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- Other objectives: minimizing shortfall risk, value at risk, etc...

The HJB equation

$u(x, s, q, t)$ solves

$$\begin{cases} u_t + \frac{1}{2}\sigma^2 u_{ss} \\ + \max_{p^b} \lambda^b(p^b) \left[u(s, x - p^b, q + 1, t) - u(s, x, q, t)\right] \\ + \max_{p^a} \lambda^a(p^a) \left[u(s, x + p^a, q - 1, t) - u(s, x, q, t)\right] = 0 \\ u(S, x, q, t) = -\exp(-\gamma(x + qS)). \end{cases}$$

The indifference or reservation prices

Definition

The indifference bid price r^b (relative to a book of q stocks) is given implicitly by the relation

$$u(x - r^b(s,q,t), s, q+1, t) = u(x, s, q, t).$$

The indifference ask price r^a solves

$$u(x + r^a(s,q,t), s, q-1, t) = u(x, s, q, t).$$

The optimal quotes

Theorem

The optimal bid and ask prices p^b and p^a are given by the implicit relations

$$p^b = r^b - \frac{1}{\gamma} \ln \left(1 + \gamma \frac{\lambda^b}{\frac{\partial \lambda^b}{\partial p}} \right)$$

and

$$p^a = r^a + \frac{1}{\gamma} \ln \left(1 - \gamma \frac{\lambda^a}{\frac{\partial \lambda^a}{\partial p}} \right).$$

The "Frozen-Inventory" Approximation

- If we assume there is no arrival of orders

$$
\begin{aligned}
v(x, s, q, t) &= E_t[-\exp(-\gamma(x + qS_T))] \\
&= -\exp(-\gamma x)\exp(-\gamma qs)\exp\left(\frac{\gamma^2 q^2 \sigma^2 (T-t)}{2}\right)
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- The indifference price of a stock, given an inventory of q stocks is

$$r(s,q,t) = s - q\gamma\sigma^2(T-t)$$

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- This is an approximation to r^a and r^b for the problem with order arrivals

The "Econophysics" Approximation

1. The density of market order size is

$$f^Q(x) \propto x^{-1-\alpha}$$

Gabaix et al. (2006)

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- Imply that arrival rates are exponential

$$\lambda^a = A\exp(-k(p^a - s)) \text{ and } \lambda^b = A\exp(-k(s - p^b))$$

The optimal quotes

- Step one: the indifference price

$$r(s, q, t) = s - q\gamma\sigma^2(T - t)$$

The optimal quotes

- Step one: the indifference price

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- Step two: the bid/ask quotes

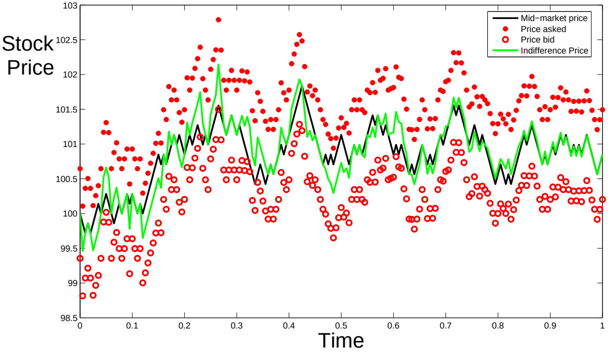
$$p^b = r - \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right)$$

and

$$p^a = r + \frac{1}{\gamma}\ln\left(1+\frac{\gamma}{k}\right).$$

k is a measure of the liquidity of the market.

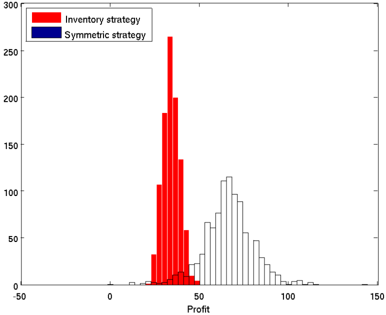
A stock price simulation for $\gamma = 0.1$



P&L profile for $\gamma = 0.5$

Strategy	Spread	Profit	std(Profit)	std(Final q)
Inventory	1.15	33.92	4.72	1.88
Symmetric	1.15	66.20	14.53	9.06

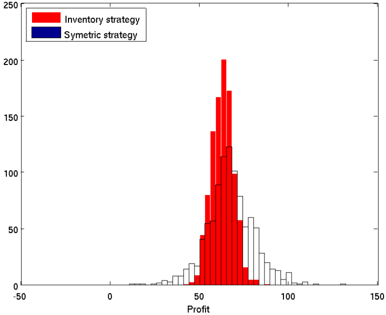
Table: 1000 simulations with $\gamma = 0.5$



P&L profile for $\gamma = 0.1$

Strategy	Spread	Profit	std(Profit)	std(Final q)
Inventory	1.29	62.94	5.89	2.80
Symmetric	1.29	67.21	13.43	8.66

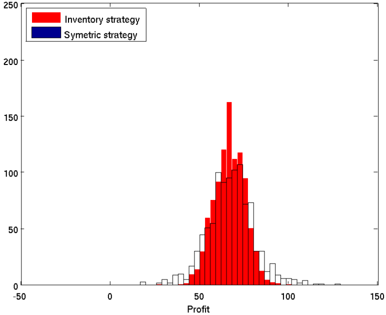
Table: 1000 simulations with $\gamma = 0.1$



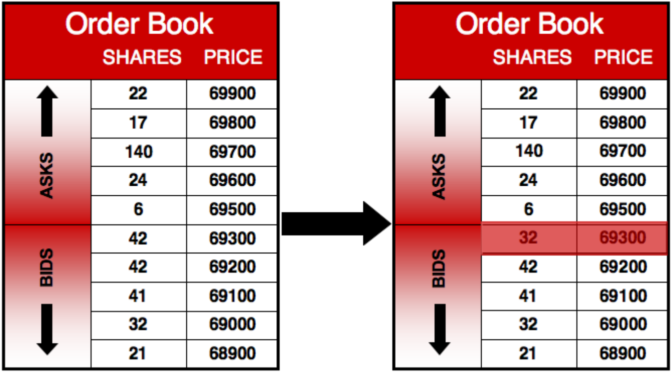
P&L profile for $\gamma = 0.01$

Strategy	Spread	Profit	std(Profit)	std(Final q)
Inventory	1.33	66.78	8.76	4.70
Symmetric	1.33	67.36	13.40	8.65

Table: 1000 simulations with $\gamma = 0.01$



A market order



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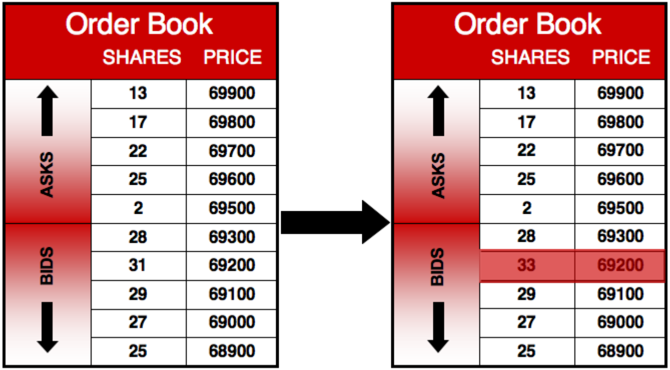

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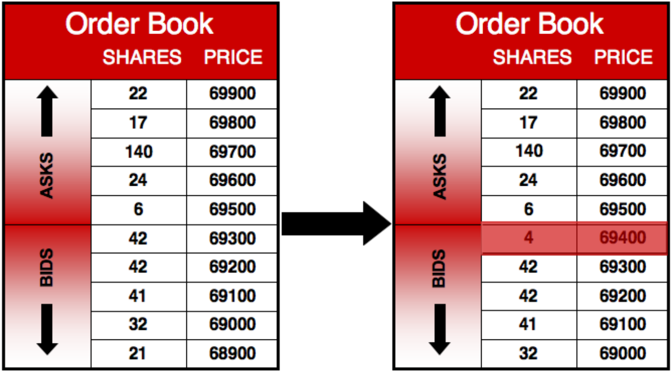

A cancellation

Order Book (before)

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ASKS	13	69900
	17	69800
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	25	69600
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BIDS	28	69300
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Order Book (after)

	SHARES	PRICE
ASKS	10	69900
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- The sizes of market and limit orders are random.
- The above events are mutually independent.

The simulation pseudocode

At each time step, generate the next event:

- Probability of a market buy order

$$\frac{\mu^a}{\mu^b + \mu^a + \sum_d (\lambda^b(d) + \lambda^a(d)) + \sum_d \theta(d) Q_t^b(d) + \sum_d \theta(d) Q_t^a(d)}$$

Draw order size (in shares) from empirical distribution.

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- Probability of a limit buy order i ticks away from the best ask

$$\frac{\lambda^b(i)}{\mu^b + \mu^a + \sum_d (\lambda^b(d) + \lambda^a(d)) + \sum_d \theta(d) Q_t^b(d) + \sum_d \theta(d) Q_t^a(d)}$$

  Draw order size from empirical distribution

The simulation pseudocode

- Probability of a cancel buy order i ticks away from the best ask

$$\frac{\theta(i)Q_t^b(i)}{\mu^b+\mu^a+\sum_d(\lambda^b(d)+\lambda^a(d))+\sum_d\theta(d)Q_t^b(d)+\sum_d\theta(d)Q_t^a(d)}$$

  If there are j orders at that price, cancel one of them with uniform probability.

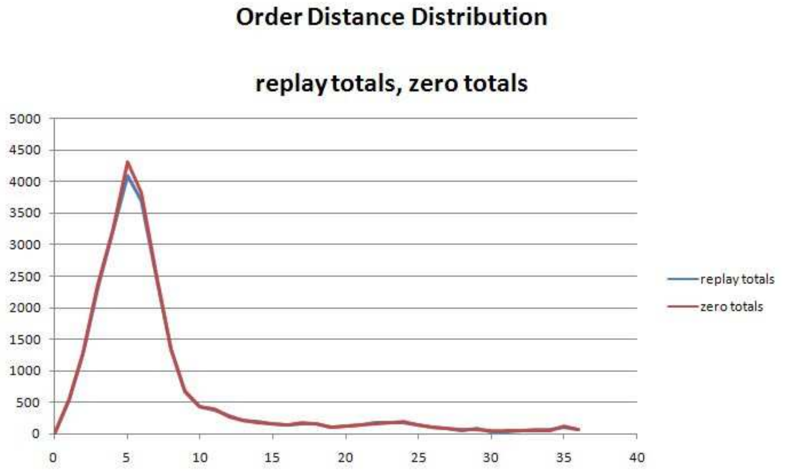
... same procedure for the sell side of the book.

The simulation parameters

- Ticker: AMZN
- Number of events (market, limit, cancel): 50.000
- Number of market orders: $\mu^a + \mu^b = 2.371$
- Number of limit orders within a 2 dollar window: $\sum \lambda^a(d) + \lambda^b(d) = 24.221$
- Number of cancel orders within a 2 dollar window: $\sum_d \theta(d) Q_t^b(d) + \sum_d \theta(d) Q_t^a(d) = 22.613$

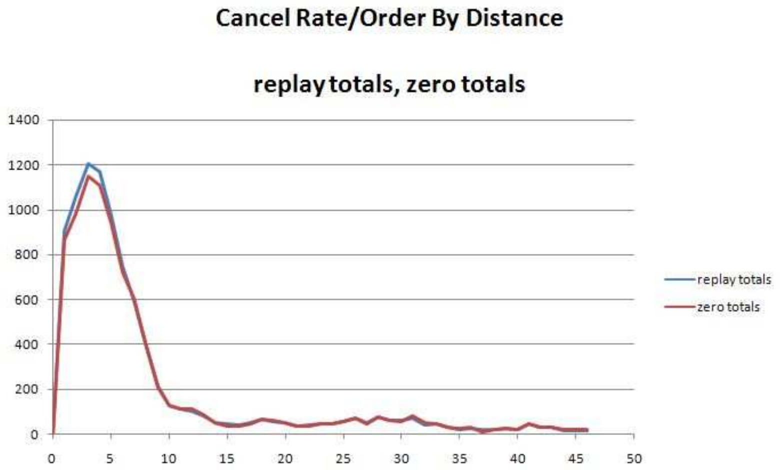
The distribution of limit orders

as a function of the distance to the opposite best quote $\lambda(d)$

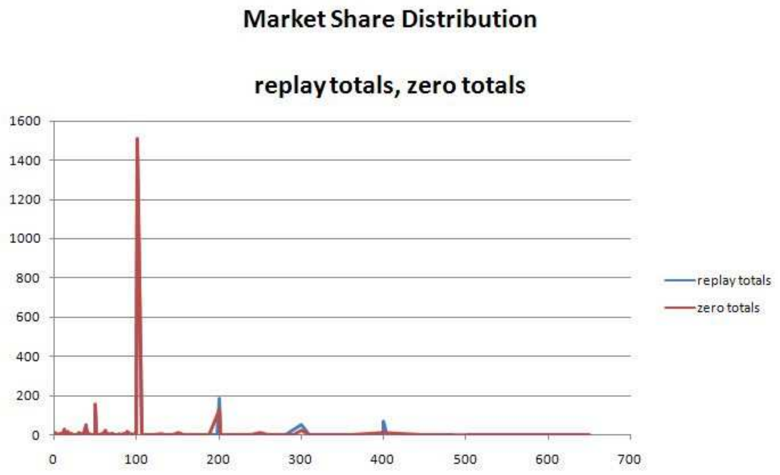


The cancel rates per order

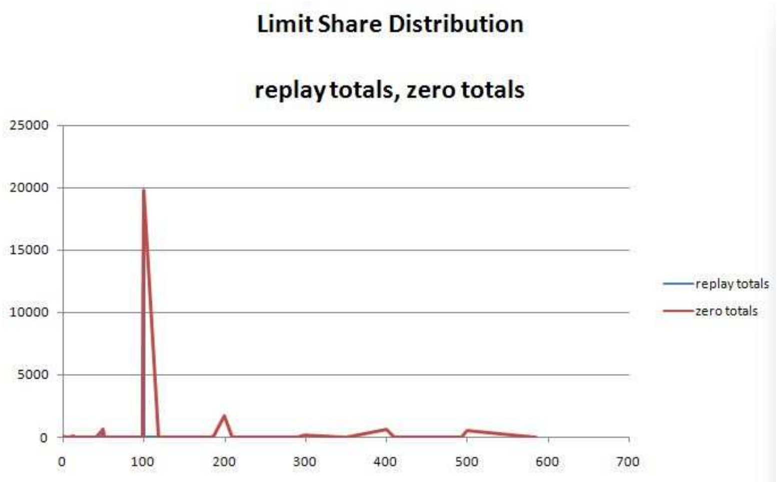
as a function of the distance to the opposite best quote $\theta(d)$



The market order size distribution

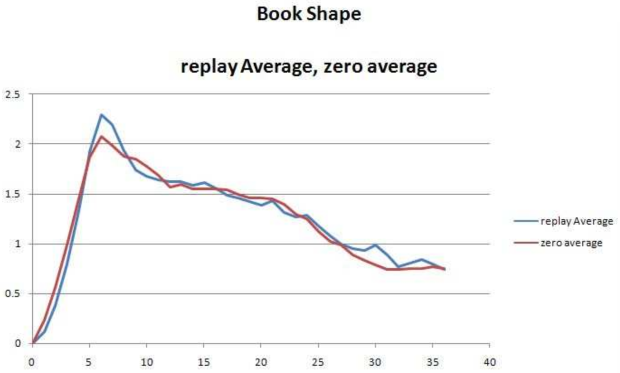


The limit order size distribution



The zero market

The average book shape



The zero market

Sample paths

Individual agent parameters

The Trump agent controls inventory by lowering the quotes after he buys, and raising the quotes after he sells. His properties include the following parameters:

- A start time (e.g. right after the book is seeded)
- A premium around the market spread (e.g. bid minus δ_b=2 cents, ask plus δ_a=2 cents)
- A position limit (e.g. 500 shares)
- A lot size (e.g. 100 shares)
- An aggressiveness parameter for inventory control

Individual agent pseudocode

Trump agent operates by:

1. Condition: If time>start time and Trump does not have two outstanding limit orders
2. The action: cancel outstanding orders and submit two limit orders at the prices

$$p_b = m_b - \delta_b + \delta_b \frac{q}{floor} * aggr$$

and

$$p_a = m_a + \delta_a - \delta_a \frac{q}{ceiling} * aggr$$

where the first term is the market bid or ask, the second term is the bid and ask premium and the third term controls the inventory. If the floor is reached, there is no ask quote. If the ceiling is reached, there is no bid quote.

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- Adverse selection loss = 3,101 $

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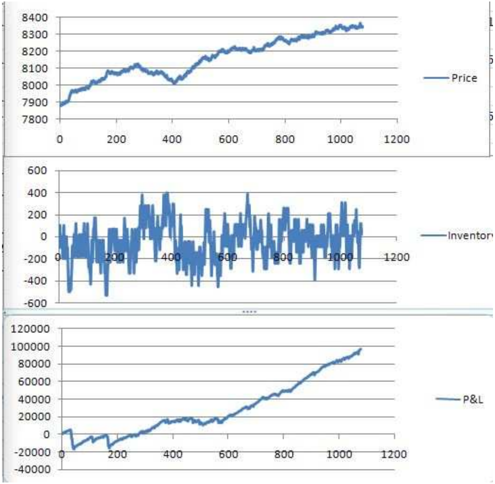

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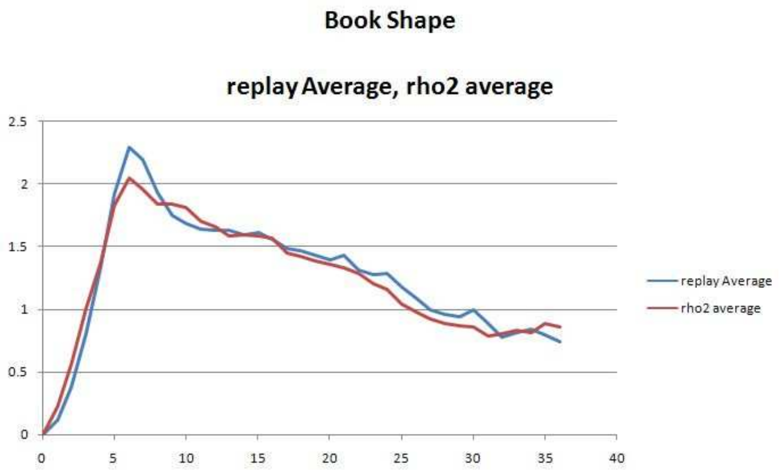
- Empirically, the market data has long sequences of BUY (resp. SELL) orders
- We implement autocorrelation:
  1. Label $X_i = 1$ for a buy order and $X_i = 0$ for a sell order
  2. Run a regression

$$X_i = \alpha + \beta_1 X_{i-1} + \ldots + \beta_{10} X_{i-10}$$

  3. In the simulation, enforce

$$P(X_i = 1) = \alpha + \beta_1 X_{i-1} + \ldots + \beta_{10} X_{i-10}$$

The rho market



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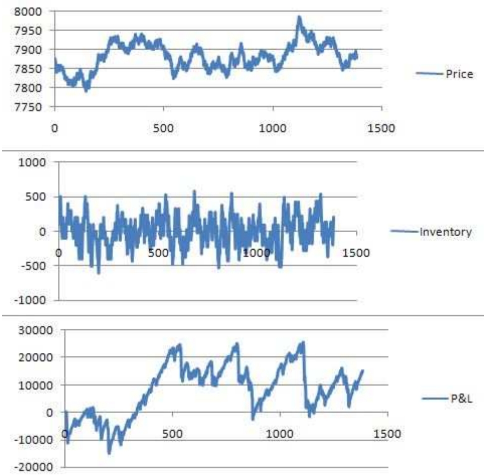
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7950
7900
7850
7800
7750
0
500
1000
1500
Price
1000
500
0
-500
-1000
0
500
1000
1500
Inventory
30000
20000
10000
0
-10000
-20000
0
500
1000
1500
P&L

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- Order book simulations:
  - We model an order book as a continuous-time Markov chain
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